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Clip-level crossing rates in photon correlation and analogue signal processing

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Abstract. An investigation is made of the rate at which the number of counts in consecutive sample times in a photon counting experiment is found to lie on either side of a fixed level. The same method is used to generalise the conventional theory of zero-crossings of an analogue signal to include finite processing bandwidth. In this case the resulting formulae are applicable to certain situations of practical importance, such as the Lorentzian spectrum, which have previously given singular results.

1. Introduction

It is well known that a randomly varying signal $V(t)$ is completely specified by a knowledge of the joint probability distributions $P(\{V(t_i)\})$. These allow moments and correlation functions of any order to be calculated. In the field of optical signal processing, early photon counting experiments investigated the simple probability distribution $p(n; T)$ of the random variable n of counts arriving in a sample time T (Johnson *et al* 1966), and more recently the measurement of more complicated statistical properties of the photon arrival rate, such as 'scaled' and 'clipped' photon correlation functions, has become more commonplace for various practical applications in spectroscopy (Cummins and Pike 1974).

One property which has not yet been investigated in the photon counting context is the quantity corresponding to the zero-crossing rate of an analogue signal. A long-standing interest in the latter stems in part from the use of various forms of zero-crossing detector in radar and communications systems (Skolnik 1962). The early results of Rice (1945) and others on the subject now appear in standard texts on communication theory (e.g. Middeton 1960). For example, the average number of zeros per second $\bar{n}_x$ of a Gaussian random signal $V(t)$ with zero mean and correlation coefficient,

$$g(\tau) = \langle V(t)V(t+\tau)\rangle / \langle V^2(t)\rangle$$

may be expressed in the form

$$\bar{n}_x = \frac{1}{\pi}\left(-\frac{d^2 g}{d\tau^2}\bigg|_{\tau=0}\right)^{1/2}. \tag{1}$$

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The question arises as to whether formulae of this type exist which may be of interest for optical systems. In this paper we shall establish formulae analogous to (1) for the rate at which the number of photon detections in time T crosses a fixed 'clip' level. Experimental data will be presented which support satisfactorily the theoretical results. Our analysis takes finite sample time and finite area effects into account for a Lorentzian spectral profile and sheds some light on the apparent breakdown of equation (1) for this type of spectrum in the conventional analysis.

2. Level-crossing formula

In a typical photon counting experiment we measure the random number of counts $n(t)$ registered by the photodetector at a time t and within a fixed sample time T. The result of such a measurement taken over many consecutive non-overlapping sample intervals yields a photon count rate such as that shown in figure 1(a). By introducing a clip level k we may define the clipped photon count rate $n_k(t)$ (Jakeman and Pike 1969) such that

$$n_k(t) = \begin{cases} 1 & \text{if } n(t) > k \\ 0 & \text{if } n(t) \leqslant k. \end{cases} \tag{2}$$

Figure 1(b) illustrates the clipped photon count rate $n_k(t)$ corresponding to the count rate $n(t)$ in figure 1(a).

Consider N instants of time $t_i (i = 1, \ldots, N)$ belonging to consecutive sample intervals. It follows from the definition of the clipped count rate $n_k(t)$ that

$$\sum_{i=1}^{N} (n_k(t_i) - n_k(t_i - T)) n_k(t_i) = N_{\text{ux}} \tag{3}$$

where N_{ux} is the number of times during N sample intervals that the count rate $n(t)$ crosses the clip level k from below. We take the average over an ensemble of identical

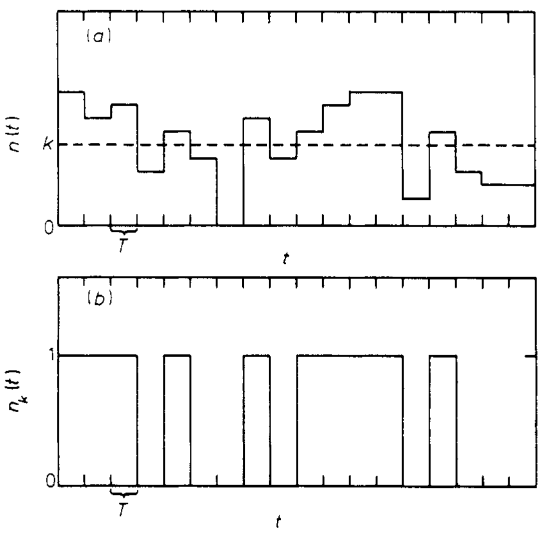

Figure 1. Typical photon count rate $n(t)$ and the corresponding clipped count rate $n_k(t)$.

experiments (noting that $n_k^2 = n_k$)

$$\sum_{i=1}^{N} (\langle n_k(t_i)\rangle - \langle n_k(t_i) n_k(t_i - T)\rangle) = \langle N_{ux}\rangle \tag{4}$$

and so, assuming stationarity, we have

$$N\langle n_k\rangle - NG_{kk}(T) = \langle N_{ux}\rangle \tag{5}$$

where

$$G_{kk}(T) = \langle n_k(0) n_k(T)\rangle \tag{6}$$

is the double-clipped photon correlation function (Jakeman and Pike 1969). Letting $N \to \infty$, equation (5) becomes

$$\bar{n}_x = 2(\langle n_k\rangle - G_{kk}(T)) \tag{7}$$

where $\bar{n}_x$ is the mean number of times during one sample interval T that the photon count rate $n(t)$ crosses the clip level k.

In order to evaluate $\bar{n}_x$ from equation (7), equation (2) is used to show that

$$\langle n_k\rangle = \sum_{n=0}^{\infty} n_k p(n; T) = 1 - \sum_{n=0}^{k} p(n; T) \tag{8}$$

and

$$\begin{aligned} \langle n_k(0) n_k(T)\rangle &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n_k m_k p(n, m; T) \\ &= 1 - 2\sum_{n=0}^{k} p(n; T) + \sum_{n=0}^{k} \sum_{m=0}^{k} p(n, m; T) \end{aligned} \tag{9}$$

where $p(n, m; T)$ is the joint probability of counting n photoelectrons in one sample interval at time t together with m photoelectrons in one sample interval at time $t + T$. Thus equation (7) may be written in the form

$$\bar{n}_x = 2\left(\sum_{n=0}^{k} p(n; T) - \sum_{n=0}^{k} \sum_{m=0}^{k} p(n, m; T)\right) \tag{10}$$

expressing the mean number of clip-level crossings per sample time in terms of the photon counting distributions.

3. Crossing rates for Gaussian–Lorentzian light

The photon counting probabilities which appear in equation (10) may be expressed in terms of their corresponding generating functions (Glauber 1963) as

$$p(n; T) = \frac{(-\alpha)^n}{n!} \frac{d^n}{ds^n} Q(s; T)\bigg|_{s=\alpha} \tag{11}$$

and

$$p(n, m; T) = \frac{(-\alpha)^n}{n!} \frac{(-\alpha)^m}{m!} \frac{d^n}{ds^n} \frac{d^m}{ds'^m} Q(s, s'; T)\bigg|_{s=s'=\alpha}. \tag{12}$$

In the case of Gaussian light with a Lorentzian spectral profile, Jakeman (1970) gives the generating function corresponding to the joint photon counting probability $p(n, m; T)$, namely

$$Q(s, s'; T) = Q_0(s, s'; T) - \tfrac{1}{3}\gamma\alpha^2\bar{n}^2 Q_0^2(s, s'; T)[s'^2 + s^2 + \alpha\bar{n}ss'(s+s')(1 - |g_{\mathrm{Lor}}^{(1)}(T)|^2)] \tag{13}$$

where

$$Q_0(s, s'; T) = [(1+\alpha\bar{n}s)(1+\alpha\bar{n}s') - \alpha^2\bar{n}^2 ss'|g_{\mathrm{Lor}}^{(1)}(T)|^2]^{-1}. \tag{14}$$

In equations (13) and (14) α is the quantum efficiency of the detector, $\bar{n}$ is the mean count rate and

$$g_{\mathrm{Lor}}^{(1)}(T) = \exp(-\Gamma T + \mathrm{i}\omega_0 T) \tag{15}$$

is the field correlation function corresponding to a Lorentzian spectrum of width Γ centred about the frequency ω_0. This generating function takes account of the small but finite sample time T and equation (13) is an expansion of the exact formula to the first order in $\gamma = \Gamma T$.

Since equations (13) and (14) are valid strictly only when the area of the detector is infinitely small, a suitable correction must also be introduced to allow for the finite detector area. Following the work of Jakeman *et al* (1971) this may be achieved by using the amended generating function

$$(\tilde{Q}(s, s'; T))^{\eta} \tag{16}$$

where the non-integral parameter η represents the number of coherence areas in the detector area and $\tilde{Q}(s, s'; T)$ is just the generating function given by equations (13) and (14), with the quantity $\alpha\bar{n}$ everywhere replaced by $\alpha\bar{n}/\eta$.

In our calculations the photon counting probabilities are evaluated analytically using the generating function (16) and the relationships (11) and (12) and thus $\bar{n}_x$ is written as a function of the basic parameters $\bar{n}$, k, γ and η. The resultant expression, although easily derived, is lengthy and involves a number of finite triple summations; it is not given here but was easily translated into a short computer program.

For small α the expression is more tractable and takes the form

$$\bar{n}_x = \frac{2}{(1+x)^{\eta}} \sum_{n=0}^{k} \frac{(\eta)_n x^n}{n!(1+x)^n} - \frac{2}{z^{\eta}} \sum_{n=0}^{k} \sum_{m=0}^{k} \left(\frac{xy}{z}\right)^{n+m} \sum_{r=0}^{[n,m]} \frac{(\eta)_{n+m-r}}{(n-r)!(m-r)!r!} \times \left(-z\frac{1-|g_{\mathrm{Lor}}^{(1)}(T)|^2}{y^2}\right)^r \tag{17}$$

where

$$x = \frac{\alpha\bar{n}}{\eta} \qquad y = 1 + x(1 - |g_{\mathrm{Lor}}^{(1)}(T)|^2) \tag{18}$$

$$z = 1 + x(1+y) \tag{19}$$

$[n, m]$ means the lesser of n and m, and $(\eta)_n$ is the Pochammer notation for $\eta(\eta+1)(\eta+2)\ldots(\eta+n-1)$.

4. Experiments

A number of experiments were carried out in order to verify these results. A Gaussian–Lorentzian light source was provided by scattering laser light from macromolecules (PSL spheres) undergoing Brownian motion in solution. A Malvern correlator system 4300 (manufactured by Malvern Instruments, Malvern, Worcs) was used to measure γ in the usual way (Blagrove *et al* 1970) by analysis of the single-clipped photon correlation function. The preset clip level k was chosen at different levels near the mean number of counts per sample time, n, which was also recorded in monitor channels of the instrument. The zero-crossing rate was measured by

Table 1. Rate of clip-level crossings $\bar{n}_x$.†

$\bar{n}$	k	γ	η	Calculated value of $\bar{n}_x$	Experimental value of $\bar{n}_x$
1·7	2	0·0217	1·0	0·212	0·229
			1·1	0·219	
			1·2	0·226	
			1·3	0·232	
6·5	7	0·0357	1·0	0·164	0·176
			1·1	0·170	
			1·2	0·176	
			1·3	0·181	
1·8	3	0·0217	1·0	0·159	0·161
			1·1	0·164	
			1·2	0·167	
			1·3	0·170	
2·5	3	0·313	1·0	0·196	0·209
			1·1	0·203	
			1·2	0·210	
			1·3	0·216	
9·6	9	0·0556	1·0	0·171	0·182
			1·1	0·176	
			1·2	0·182	
			1·3	0·187	
6·4	9	0·0357	1·0	0·136	0·142
			1·1	0·140	
			1·2	0·144	
			1·3	0·148	
6·4	8	0·0357	1·0	0·149	0·158
			1·1	0·154	
			1·2	0·159	
			1·3	0·164	

† $\bar{n}$ is the mean photon count-rate; k is the clip level; $\gamma = \Gamma T$, where T is sample time and Γ is Lorentzian spectral width; η is the number of coherence areas of the detector.

accumulating the shift register output in a further monitor channel of the correlator. The value of η due to spatial averaging by the detector can easily be measured geometrically; it can also be found by measuring the intercept on the zero delay time axis of the correlation function (Jakeman *et al* 1971, equations (12) and (14)). In the present experiments a value of $\eta = 1{\cdot}2$ was used. In order to illustrate the dependence of $\bar{n}_x$ on η computations were performed at different values of η between 1·0 and 1·3 as shown in table 1. At the value of $\eta = 1{\cdot}2$, good agreement is seen to be obtained between the calculated and measured values of $\bar{n}_x$ over the entire range of parameters chosen in this set of experiments giving an independent check on the procedures adopted.

5. Dependence on clip level and detector area

Further calculations were carried out to study the dependence of $\bar{n}_x$ on both the clip level k and the detector area parameter η. In these calculations the value of $\gamma = \Gamma T$ was small and the contribution from the second term in equation (13) was found not to be important.

The behaviour of $\bar{n}_x$ as a function of the clip level k is plotted in figure 2 for different values of the mean count rate $\bar{n}$, where we have taken the area parameter $\eta = 1{\cdot}0$ and $\gamma = 0{\cdot}01$. For small values of $\bar{n}$, the probability of obtaining more than one count in each sample time is low and so the number of clip-level crossings falls off rapidly as the clipping level is increased from 0. However at $\bar{n} = 10$, where the probability of obtaining few counts per sample time is small, the number of clip-level crossings rises as the clipping level k is increased from 0 and reaches a maximum when $k \simeq 5$.

In figure 3 the number of clip-level crossings per sample time is plotted as a function of the detector area parameter for different values of $\bar{n}/\eta$, the mean count rate per

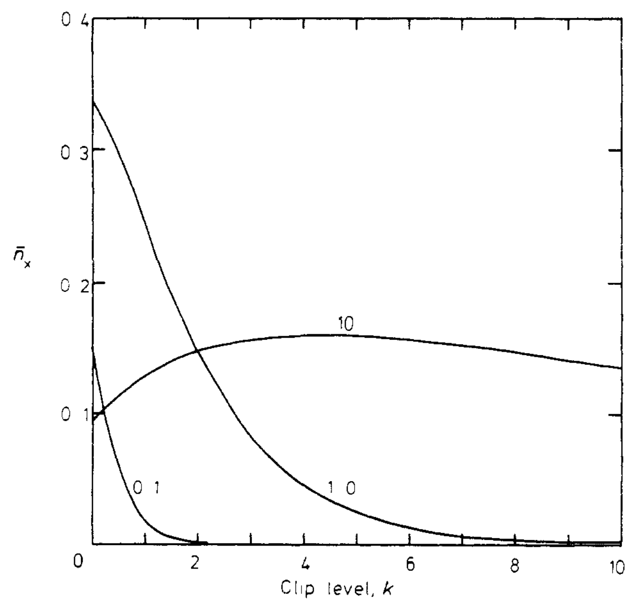

Figure 2. Number of clip-level crossings per sample time $\bar{n}_x$ as a function of the clip level k. $\gamma = 0{\cdot}01$, $\eta = 1{\cdot}0$; the curves are labelled with values of $\bar{n}$.

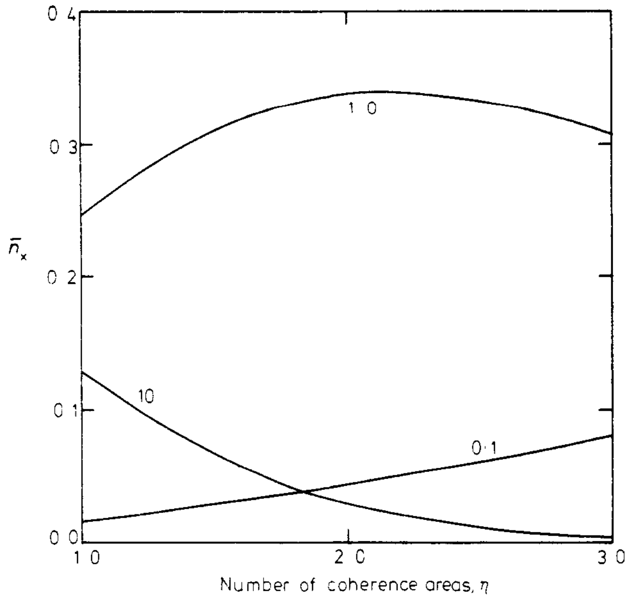

Figure 3. Number of clip-level crossings per sample time $\bar{n}_x$ as a function of the detector area parameter η. Clip level $k = 1$, $\gamma = 0{\cdot}01$; the curves are labelled with values of $\bar{n}/\eta$.

coherence area of the detector. The clip level is chosen to be 1 with $\gamma = 0{\cdot}01$. When $\bar{n}/\eta$ is small, increasing the area of the detector causes the mean count rate $\bar{n}$ of the experiment to increase and so the number of clip-level crossings is seen to rise. When $\bar{n}/\eta = 1{\cdot}0$, $\bar{n}_x$ rises as η is increased from $1{\cdot}0$ but at a value of $\eta \simeq 2{\cdot}1$, $\bar{n}_x$ reaches a maximum before starting to decline. In this case, where the value of $\bar{n}/\eta$ is larger, not only does the increase in area of the detector increase the mean count rate $\bar{n}$ of the experiment but, in addition, the signal fluctuations begin to average out thus causing the number of clip-level crossings to decrease. When $\bar{n}/\eta$ is large the effect of averaging out information in the signal is dominant and $\bar{n}_x$ decreases as η is increased from $1{\cdot}0$.

6. Zero-crossing formula for analogue signals

A similar approach to that taken in §2 may be employed to investigate the expected rate at which a Gaussian distributed random signal $V(t)$ of zero mean crosses zero. Thus, implicit in this analysis is a finite time of measurement T which must exist in any real experimental situation and which, in certain cases, has important consequences for the results. The signal $V(t)$ is sampled in consecutive finite time intervals T. In each interval $[nT, (n+1)T]$ the value at nT is measured and thus the clipped signal $V_0(t)$ is obtained (see figure 4) where we define

$$V_0(t) = \begin{cases} +1 & \text{if } V(nT) > 0 \\ -1 & \text{if } V(nT) < 0 \end{cases} \tag{20}$$

For N instants of time $t_i (i = 1, \ldots, N)$ belonging to consecutive intervals T it follows from this definition that

$$\sum_{i=1}^{N} V_0(t_i)(V_0(t_i) - V_0(t_i - T)) = 2N_x \tag{21}$$

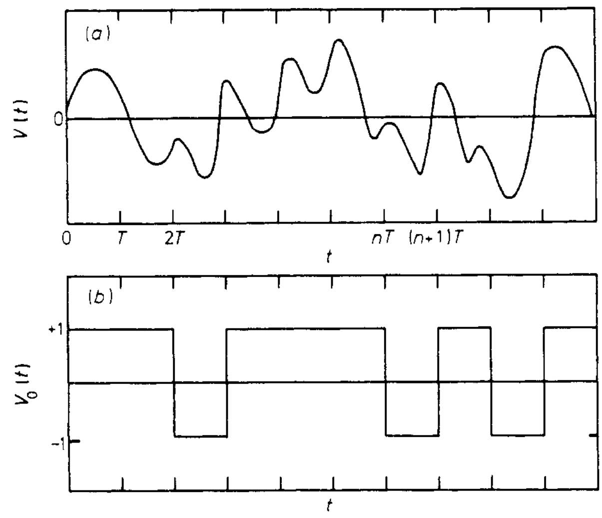

Figure 4. Random signal $V(t)$ and the corresponding clipped signal $V_0(t)$.

where N_x is the number of times during N sample intervals that the measured signal crosses zero. Assuming stationarity and taking N to be large it follows that

$$\bar{n}_x = \tfrac{1}{2}(\langle V_0^2 \rangle - G_{00}(T)) \tag{22}$$

where $\bar{n}_x$ is the mean number of times that the measured signal crosses zero during one sample time,

$$G_{00}(T) = \langle V_0(t) V_0(t+T) \rangle \tag{23}$$

and the angle brackets, as before, denote the average over a large ensemble of identical experiments. Now from equation (20)

$$G_{00}(T) = \langle V_0(t) V_0(t+T) \rangle = \langle V_0(nT) V_0((n+1)T) \rangle \tag{24}$$

and, since nT, $(n+1)T$ are points on which $V(t)$ has been measured, it follows from the theorem of Van Vleck and Middleton (1966) that

$$G_{00}(T) = \frac{2}{\pi} \sin^{-1}(g(T)) \tag{25}$$

where

$$g(T) = \langle V(t) V(t+T) \rangle / \langle V^2(t) \rangle. \tag{26}$$

Substituting this relationship into equation (22), and noting from equation (20) that $\langle V_0^2 \rangle = 1$, we find that the expected number of zero-crossings per unit time is given by

$$\frac{\bar{n}_x}{T} = \frac{1}{2T}\left(1 - \frac{2}{\pi} \sin^{-1}(g(T))\right) = \frac{1}{\pi T} \sin^{-1}[(1-g^2(T))^{1/2}]. \tag{27}$$

As the sample time $T \to 0$, $g(T) \to 1$ and so

$$\begin{aligned} &\sin^{-1}[(1-g^2(T))^{1/2}] \\ &\quad \simeq (1-g^2(T))^{1/2}[1 + \tfrac{1}{6}(1-g^2(T))] \\ &\quad = (-2Tg'(0) + \tfrac{1}{3}T^2 g'(0)^2 - T^2 g''(0))^{1/2}. \end{aligned} \tag{28}$$

Therefore

$$\frac{\bar{n}_x}{T} \simeq \frac{1}{\pi}\left(-\frac{2g'(0)}{T} + \frac{1}{3}g'(0)^2 - g''(0)\right)^{1/2} \tag{29}$$

and so as $T \to 0$ the expected rate of zero-crossings is finite only if $g'(0) = 0$, in which case

$$\frac{\bar{n}_x}{T} = \frac{1}{\pi}(-g''(0))^{1/2}. \tag{30}$$

This result, which holds only when $g''(0)$ is negative, is identical to that obtained by Rice (1945).

We now show that the condition $g'(0) = 0$ is equivalent to requiring the mean squared value $\langle V'(t)^2\rangle$ of the slope of a random signal to be finite. Since

$$V'(t) = \lim_{T\to 0} \frac{V(t+T) - V(t)}{T} \tag{31}$$

then

$$\langle V'(t)^2\rangle = \lim_{T\to 0}\left\langle\frac{(V(t+T) - V(t))^2}{T^2}\right\rangle, \tag{32}$$

i.e.

$$\langle V'(t)^2\rangle = 2\langle V^2\rangle \lim_{T\to 0}\left(\frac{1-g(T)}{T^2}\right). \tag{33}$$

Hence, since $g(T) \to 1$ as $T \to 0$, $\langle V'(t)^2\rangle$ is finite only if, for small T, the quantity $1 - g(T)$ is of at least second order in T. Thus $(1 - g(T))/T$ is of at least first order in T which implies that $g'(0) = 0$.

Rice (1945) has pointed out that equation (30) does not hold for certain important spectra such as the Lorentzian, in which case $g'(0)$ is negative and $g''(0)$ is positive. However, if the effect of finite sample time T is included, equation (26) may be applied and we see that for small T

$$\frac{\bar{n}_x}{T} \simeq \frac{1}{\pi}\left(-\frac{2g'(0)}{T}\right)^{1/2}. \tag{34}$$

This holds when $g'(0)$ is negative, as for a Lorentzian spectrum, giving a large but finite number of zero crossings. Thus if

$$\omega(f) = \frac{a}{f^2 + a^2} \tag{35}$$

so that

$$g(T) = \mathrm{e}^{-2\pi a T} \tag{36}$$

then $g'(0) = -2\pi a$ and so the expected rate of zero crossings is given by

$$\frac{\bar{n}_x}{T} = 2\left(\frac{a}{\pi T}\right)^{1/2}. \tag{37}$$

This result also follows from Rice's analysis when it is generalised to take account of a small but finite time of measurement. In spectral terms, a finite time of measurement T

corresponds to an upper frequency F which is the Nyquist frequency (Kendall and Stuart 1966) given by

$$F = 1/2T. \tag{38}$$

Inserting, therefore, this finite upper frequency, Rice's expression (equation (3.3–11)) for the expected number of zero crossings per second becomes

$$\frac{\bar{n}_x}{T} = 2\left(\frac{\int_0^F f^2\omega(f)\,\mathrm{d}f}{\int_0^F \omega(f)\,\mathrm{d}f}\right)^{1/2} \tag{39}$$

which may be integrated using equation (35) to give

$$\frac{\bar{n}_x}{T} = 2\left(\frac{aF}{\tan^{-1}(F/a)} - a^2\right)^{1/2} \simeq 2\left(\frac{2aF}{\pi}\right)^{1/2} \tag{40}$$

since F is large. In terms of the sample time T we have

$$\frac{\bar{n}_x}{T} = 2\left(\frac{a}{\pi T}\right)^{1/2} \tag{41}$$

which is the result obtained in equation (37).

Since real correlation functions must have zero slope at the origin in order to ensure causal behaviour (Rice 1945), it is not possible in theory to have a Gaussian distributed random signal with a completely Lorentzian spectral profile. However, in many physical experiments an apparent behaviour of this kind is observed due to the finite response time of the apparatus. It is never physically possible to measure the slope of a real correlation function exactly at the origin since $\lim_{\tau\to 0}(g(\tau)-g(0))/\tau$ cannot be attained due to the existence of a finite integration time which bounds τ from below, and if this time is sufficiently long compared with the duration of reversible processes, then these apparently non-physical correlation functions are obtained. In such cases the number of zero crossings per second will be given by equation (29) above.

7. Conclusions

We have derived general expressions for clip-level crossing rates of Gaussian signals, both analogue and digital, taking account of finite detection integration time. Simple circuits which measure these crossing rates can, therefore, be used to measure the linewidth of signals of known spectral shape. It is interesting to note that in the case of a spectrum consisting of a sum of Lorentzians, the slope $g'(0)$ of the correlation function at the origin, which may be obtained from the crossing rate (equation (34) or equation (7) in the limit $T \to 0$), is of direct physical interest as a useful weighted mean linewidth (see, for example, Pusey 1977).

In the case of optical signals the relation (7) shows that the information contained in $\bar{n}_x$ can also be obtained by simultaneous measurement of $G_{kk}(T)$ and $\langle n_k \rangle$ which is experimentally of the same order of complexity. The first two channels of a standard single-clipping correlator or even the second moment of a photon counting distribution provide similar information. The relative accuracies of linewidth measurements made by any of these methods might be expected to be of comparable magnitude.

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